Chapter 5

ENVELOPE EQUATION

We often read that when the linear part of the space charge force is added to the linear equation of motion, it produces an incoherent tune shift, which if large enough can place individual particles onto low-order betatron resonant lines resulting in an instability. This picture, although appealing, is very misleading. In fact, the resonant driving force drives the beam to resonance only when the *coherent* space charge tune shift lands the coherent betatron tune of the beam onto the resonance lines. We are going to show that resonant driving force of any order will *not* affect an individual particle when the space charge force shifts its betatron tune onto the resonance line of that order.

5.1 The Integer Resonance

In this section, we are going to study the effects on beam particles under the influence of errors in the dipoles. We will find that although the beam center is able to see the force from the dipole errors, it will not see the self-fields from the beam particles. On the other hand, a single particle sees the self-fields and has its betatron tunes shifted. However, a single particle oscillating at an integer tune will be not be driven by the dipole-error force. We shall follow a discussion by Baartman [1].

The integer resonance is driven by errors in the dipoles around the accelerator ring. The transverse motion of a beam particle is governed by

$$\frac{d^2X}{d\psi^2} + \nu_{0x}^2 X = F_{sc\,x} + F_{e\,x}(\psi) \tag{5.1}$$

where

$$\psi = \int_0^s \frac{ds}{\nu_{0x}\beta_x(s)} \tag{5.2}$$

is the transverse Floquet phase which advances by 2π per turn, X is the normalized transverse offset (actual offset x divided by square root of the betatron function β_x), and ν_{0x} is the bare betatron tune. The *force** due to errors in dipoles in the x-direction is represented by $F_{e\,x}(\psi)$, which is periodic in ψ and is X independent. The space charge force $F_{sc\,x}$, if linear, can be written as

$$F_{sc\,x} = -2\nu_{0x}\Delta\nu_{sc}(X - \langle X\rangle) \ , \tag{5.3}$$

where $\langle X\rangle$ is the transverse offset of the center of charge of the beam and $\Delta\nu_{sc}$ is the incoherent space charge tune shift depicted in, for example, Eq. (4.24). The equation of motion becomes

$$\frac{d^2X}{d\psi^2} + \nu_{0x}^2 X = -2\nu_{0x}\Delta\nu_{sc}(X - \langle X\rangle) + F_{e\,x}(\psi) \ . \tag{5.4}$$

Taking the average, we obtain the equation of motion for the center of the beam,

$$\frac{d^2\langle X\rangle}{d\psi^2} + \nu_{0x}^2\langle X\rangle = F_{e\,x}(\psi) \ . \tag{5.5}$$

The space charge term disappears, indicating that the motion of the center of charge is not affected by the space charge self-force. Physically, the beam transverse motion is rigid and therefore the center cannot see any change in the pattern of the space charge self-field. In other words, there is no coherent dipole space charge tune shift. However, we do see that the center of the beam is driven by the dipole force due to lattice error. The beam will be unstable if the coherent tune ν_{0x}, or just bare tune here, is equal to an integer. Another way of saying the same thing is that as the coherent tune approaches an integer, the closed-orbit distortion, being kicked in the same direction in every turn, increases without limit. To show this more clearly, let us write the nth-harmonic component of the periodic lattice-error force as $F_{e\,x}(\psi) = f_n e^{in\psi}$. The particular solution of Eq. (5.5) is

$$\langle X\rangle = \frac{f_n e^{in\psi}}{\nu_{0x}^2 - n^2} \ , \tag{5.6}$$

*Here $F_{sc\,x}$ and $F_{e\,x}$ do not have the dimension of a force. They should be forces divided by appropriate variables. But for simplicity, we just call them forces.

which is indeed unstable when the $\nu_{0x} = n$.

The incoherent motion is obtained by subtracting Eq. (5.5) from Eq. (5.4),

$$\frac{d^2}{d\psi^2}\left(X - \langle X\rangle\right) + \left(\nu_{0x}^2 + 2\nu_{0x}\Delta\nu_{sc}\right)\left(X - \langle X\rangle\right) = 0 \ , \tag{5.7}$$

showing that an individual particle is making betatron motion about the center of the beam with the incoherent betatron tune $\nu_{\rm incoh} = \nu_{0s} + \Delta\nu_{sc}$. It is important to notice that the incoherent equation of motion contains no driving terms for the integer resonance. Therefore, incoherent motion is not affected by dipole errors. This means that the incoherent tune can be equal to an integer with no adverse effects. It is worth re-emphasizing: A particle which is shifted by direct space charge to a tune of exactly an integer, turn by turn sees the same dipole errors at the same betatron phase, and yet is not even slightly affected compared with other particles which do not have an integer tune. This is not due to space charge stabilizing the resonance, as claimed by Ref. [2], because in this example of linear space charge, there is no incoherent tune spread to generate Landau damping. The correct answer is simply no driving term for incoherent motion.

This concept can be extended easily to any nonlinear space charge force. For the ith particle, the equation of motion is

$$\frac{d^2X_i}{d\psi^2} + \nu_{0x}^2 X_i = \sum_j{}' F_{ij} + F_{ex} \ , \tag{5.8}$$

where F_{ij} is the force of the jth particle acting on the ith particle, and $\sum_j'$ implies a summation over j but with $j = i$ excluded. Thus, $\sum_j' F_{ij}$ is just the space charge force on the ith particle. We now take the average of Eq. (5.8) by summing over i, giving exactly Eq. (5.5) again. This result is obtained because of Newton's third law: $F_{ij} = -F_{ji}$ when $i \neq j$. Subtracting Eq. (5.5) from Eq. (5.8), we arrive at the incoherent equation

$$\frac{d^2}{d\psi^2}\left(X - \langle X\rangle\right) + \nu_{0x}^2\left(X_i - \langle X\rangle\right) = \sum_j{}' F_{ij} \ . \tag{5.9}$$

Again, there is no dipole driving force for the equation of incoherent motion. The space charge self-force, being nonlinear, does not just reduce to a simple incoherent tune shift. The incoherent tune will be different for different particle depending on its amplitude and the transverse beam distribution. However, whatever the incoherent tune is, even at an integer, the individual particle will not be affected by the dipole lattice error at all.

5.2 The K-V Equation

Now let us come to the errors in the quadrupoles. This force, denoted by $XF(\psi)$ is responsible for the half-integer resonance. Sometimes it is also called the linear error force, because quadrupoles are linear elements of the accelerator lattice. The equation of transverse motion for a particle is

$$\frac{d^2X}{d\psi^2} + \nu_{0x}^2 X = -2\nu_{0x}\Delta\nu_{sc}(X - \langle X\rangle) + XF(\psi) \ , \tag{5.10}$$

where a linear space charge force $-2\nu_{0x}\Delta\nu_{sc}(X - \langle X\rangle)$ has been assumed. Coherent motion is obtained by averaging Eq. (5.10),

$$\frac{d^2\langle X\rangle}{d\psi^2} + \nu_{0x}^2\langle X\rangle = \langle X\rangle F(\psi) \ , \tag{5.11}$$

and the difference gives the incoherent motion

$$\frac{d^2}{d\psi^2}(X - \langle X\rangle) + \left(\nu_{0x}^2 + 2\nu_{0x}\Delta\nu_{sc}\right)(X - \langle X\rangle) = (X - \langle X\rangle)\,F(\psi) \ . \tag{5.12}$$

It appears in Eq. (5.12) that the incoherent motion is driven by the quadrupole-error force so that the particle will experience an instability at the half integer. This conclusion is incorrect, although there is nothing wrong with the derivation from Eqs. (5.10) to (5.12). A quadrupole in the lattice will change the size of the particle beam and so will the quadrupole-error force. The incoherent space charge tune shift depends on the beam size, which is a function of the quadrupole error force $XF(\psi)$. Actually, the effect of the quadrupole-error force inside the incoherent space charge tune shift just cancels the quadrupole-error force on the right side of Eq. (5.12), leaving behind an incoherent motion not affected by the quadrupole-error force. To demonstrate this, we need to study the equation of motion governing the beam size or beam envelope.

The dipole coherent tune shifts are zero because the beam center does not experience any variation of the forces between beam particles, when the beam is executing rigid dipole oscillations as a whole. Thus, the space charge forces do not affect the restoring force of rigid oscillation and therefore do not affect the dipole coherent tunes. However, there are other collective modes of oscillation in a beam. Examples are the breathing mode, where the transverse beam size expands and contract without the beam center being moved, and the mode when the breathing in the two transverse directions are 180°

out of phase. The restoring forces of these modes of oscillation do depend on the forces between the beam particles. Thus, their frequencies of oscillation are affected by the space charge forces. To study these modes, we need to resort to the equations of motion governing the beam envelope.

The envelope equation was first derived by Kapchinsky and Vladimirsky [3] for a coasting beam with uniform charge density and elliptical cross section. Later it was generalized by Sacherer [4] to include any distribution when the beam envelope $\hat{x}$ is replaced by the rms beam size $\tilde{x} = \sqrt{\langle x^2 \rangle}$ of the beam. We are going to follow Sacherer's approach.

Consider an ensemble of particles that obey the single-particle equations

$$\begin{aligned} x' &= p_x \ , \\ p_x' &= F_x(x, s) \ , \end{aligned} \tag{5.13}$$

where x is the transverse offset, p is the canonical momentum, and the prime denotes derivative with respect to time s, the distance along the designed orbit of the accelerator. The total force† in the x-direction,

$$F_x(x, s) = F_{sc\,x} + F_{\mathrm{ext}\,x} \ , \tag{5.14}$$

includes the space charge self-force $F_{sc\,x}$ and the external force $F_{\mathrm{ext}\,x}$. Averaging over the particle distribution $f(x, p, s)$, we obtain the equations of motion for the center of the beam:

$$\begin{aligned} \langle x \rangle' &= \langle p_x \rangle \ , \\ \langle p_x \rangle' &= \langle F_x(x, s) \rangle = \langle F_{\mathrm{ext}\,x} \rangle \ . \end{aligned} \tag{5.15}$$

where the last equation follows from $\langle F_{sc\,x} \rangle = 0$ because of Newton's third law. Note that the order of the averaging and differentiation with respect to s is immaterial and can be interchanged if one wishes. For a linear machine, for example with only dipoles and quadrupoles in the ring, the external force is linear. We can write $F_{\mathrm{ext}\,x} = K_x(s)x$, and the equation of motion governing the center of the beam becomes

$$\langle x \rangle'' + K_x(s) \langle x \rangle = 0 \ , \tag{5.16}$$

which involves only first moments and is independent of the space charge force or the detailed form of the beam distribution.

†We call them forces, although $F_x(x, s)$, $F_{sc\,x}$, and $F_{\mathrm{ext}\,x}$ do not have the dimension of a force. Note that they have different dimension from the forces introduced in Eq. (5.1).

The second moments satisfy the equations

$$\begin{aligned}
\langle x^2\rangle' &= 2\,\langle xx'\rangle = 2\,\langle xp_x\rangle\ , \\
\langle xp_x\rangle' &= \langle x'p_x\rangle + \langle xp_x'\rangle = \langle p_x^2\rangle - K_x(s)\langle x^2\rangle + \langle xF_{scx}\rangle\ , \\
\langle p_x^2\rangle' &= 2\,\langle pp_x'\rangle = -2K_x(s)\langle xp_x\rangle + 2\,\langle p_xF_{scx}\rangle\ .
\end{aligned} \tag{5.17}$$

Notice that this set of equations is not closed because $\langle xF_{scx}\rangle$ and $\langle pF_{scx}\rangle$ are usually functions of the higher moments like $\langle x^n\rangle$, $\langle x^np_x\rangle$, etc. As will be demonstrated below, if the self-force is derived from the free-space Poisson equation, $\langle xF_{scx}\rangle$ depends mainly on the second moments and very little, if at all, on the higher moments.

Let us introduce the rms emittance

$$E_x = \sqrt{\langle x^2\rangle\ \langle p_x^2\rangle - \langle xp_x\rangle^2}\ . \tag{5.18}$$

Using the rate of change in the second moments in Eq. (5.17), the rate of change of the rms emittance along the accelerator is

$$E_x' = \frac{\langle xp_x\rangle\,\langle xF_{scx}\rangle - \langle x^2\rangle\,\langle p_xF_{scx}\rangle}{E_x}\ . \tag{5.19}$$

Thus, the rms emittance is an invariant provided that the space charge force is linear, or when it can be written as $F_{scx} = \epsilon(s)\,(x - \langle x\rangle)$. However, if we assume that the rms emittance is either time invariant or its time dependency is known *a priori*, $\langle p_x^2\rangle$ can be expressed in terms of $\langle x^2\rangle$, $\langle xp_x\rangle$, and E_x. Then, the first two equations in Eq. (5.17) form a closed set, and they can be combined to give

$$\tilde{x}'' + K(s)\tilde{x} - \frac{E_x^2}{\tilde{x}^3} - \frac{\langle xF_{scx}\rangle}{\tilde{x}} = 0\ , \tag{5.20}$$

where $\tilde{x} = \sqrt{\langle x^2\rangle}$ is the rms beam size.

The space charge term has an interesting interpretation. If we define the linear part of the space charge force F_{scx} as $\varepsilon(s)x$, where $\varepsilon(s)$ is determined by minimizing at a fixed time

$$D = \int [\varepsilon(s)x - F_{scx}]^2\, n(x,s)dx\ , \tag{5.21}$$

where the linear distribution is

$$n(x,s) = \int f(x,p,s)dp\ , \tag{5.22}$$

and the phase-space distribution is $f(x,p_x,s)$, then we obtain

$$\varepsilon(s)x = \frac{\langle xF_{sc\,x}\rangle}{\tilde{x}^2}x \ . \tag{5.23}$$

In other words, the rms envelope equation depends only on the linear part of the space charge force determined by least square.

Finally, let us express the envelope equation in terms of the static electric field $\mathcal{E}_x$ of the space charge self-force in the x-direction and put back all the missing factors. The envelope equation in the mks units now reads

$$\tilde{x}'' + K(s)\tilde{x} - \frac{E_x^2}{\tilde{x}^3} - \frac{e}{m\gamma^3\beta^2c^2}\frac{\langle x\mathcal{E}_x\rangle}{\tilde{x}} = 0 \ , \tag{5.24}$$

where m is the mass of the beam particles. In the denominator, we have the Lorentz factor $\gamma\beta^2$ because of Newton's second law and the other γ^2 because of the presence of the magnetic field of the beam in the laboratory frame, as demonstrated in Eqs. (4.23) and (4.24).

5.2.1 One Dimension

Consider a very long beam traveling in the z-direction with very wide width in the y-direction. The situation can be approximated by a one-dimensional beam having space charge self-force only in the x-direction and we assume that its distribution is symmetric with respect to the $x=0$ plane. The static electric field $\mathcal{E}_x$ in the x-direction is given by Poisson equation

$$\frac{\partial\mathcal{E}_x}{\partial x} = \frac{e}{\epsilon_0}\,n(x,s) \ , \tag{5.25}$$

from which

$$\mathcal{E}_x = \frac{e}{\epsilon_0}\int_0^x n(x',s)dx' \ . \tag{5.26}$$

Here, $n(x,s)$ is the particle distribution per unit volume. Therefore, when integrated over x from $-\infty$ to $+\infty$, it is normalized to σ, the particle density per unit area in the y-z plane. Since the electric field is proportional to the fraction of particles it encloses between $\pm x$, we must have $\mathcal{E}_x \propto 1/\tilde{x}$. Thus,

$$\frac{\langle x\mathcal{E}_x\rangle}{\tilde{x}} = \frac{e}{\epsilon_0}\frac{\displaystyle\int_{-\infty}^{\infty} x\,\frac{n(x)}{\sigma}dx\int_0^x n(x')dx'}{\left[\displaystyle\int_{-\infty}^{\infty} x^2\frac{n(x)}{\sigma}dx\right]^{1/2}} = \frac{e\sigma}{2\epsilon_0}\,\varrho \ , \tag{5.27}$$

where we have defined the dimensionless parameter

$$\varrho = \frac{2\int_{-\infty}^{\infty} xh(x)dx \int_0^x h(x')dx'}{\left[\int_{-\infty}^{\infty} x^2 h(x)dx\right]^{1/2}} \ . \tag{5.28}$$

We have introduced a new distribution function $h(x) = n(x)/\sigma$ so that σ, the particle number per unit area in the y-z plane, is factored out and $h(x)$ is normalized to unity. It is important to point out that while ϱ is dimensionless, $h(x)$ can be scaled to anything that is convenient. For example, in a uniform distribution, we can choose the edges as ± 1, and in a Gaussian distribution, we can choose the rms spread as 1. Substituting in Eq. (5.24), the one dimensional envelope equation now becomes

$$\tilde{x}'' + K(s)\tilde{x} - \frac{E_x^2}{\tilde{x}^3} - \frac{2\pi r_0 \sigma}{\gamma^3 \beta^2}\varrho = 0 \ , \tag{5.29}$$

where $r_0 = e^2/(4\pi\epsilon_0 mc^2)$ is the classical radius of the beam particles. Table 5.1 shows the values of ϱ for four distributions. We see that for a wide range of distributions that are likely to be encountered, the variation of ϱ is less then 2.3%. Thus the one dimension rms envelope equation will be accurately described by Eq. (5.29) with $\varrho = 1/\sqrt{3}$.

Table 5.1: The values of the dimensionless parameter ϱ for a wide range of distributions. They are all close to $1/\sqrt{3}$.

Distribution	$h(x)$	$\sqrt{3}\varrho$
Uniform	$\begin{cases} \frac{1}{2} & \lvert x\rvert \le 1 \\ 0 & \lvert x\rvert > 1 \end{cases}$	1.000
Parabolic	$\begin{cases} \frac{3}{4}(1-x^2) & \lvert x\rvert \le 1 \\ 0 & \lvert x\rvert > 1 \end{cases}$	0.996
Gaussian	$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	0.977
Hollow	$\frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}$	0.987

For a uniform distribution in one dimension, the half widths of the beam is $\hat{x} = \sqrt{3}\tilde{x}$. The full emittance is $\epsilon_x = 3E_x$, since we also have $\hat{p}_x = \sqrt{3}\sqrt{\langle p_x^2 \rangle}$. The envelope equation for the half beam width in one dimension becomes

$$\hat{x}'' + K(s)\hat{x} - \frac{\epsilon_x^2}{\hat{x}^3} - \frac{2\pi r_0 \sigma}{\gamma^3 \beta^2} = 0 \ , \tag{5.30}$$

where $\varrho = 1/\sqrt{3}$ has been substituted.

5.2.2 Two Dimensions

With the absence of cross-correlations and coupling terms, the rms envelope equations in the two transverse directions are given by Eq. (5.24) and the two space charge terms $\langle xF_{scx}\rangle$ and $\langle yF_{scy}\rangle$ depend on the particle distribution. It will be shown below that $\langle xF_{scx}\rangle$ and $\langle yF_{scy}\rangle$ depend only on second moments if the distribution has the elliptical symmetry

$$n(x,y,s) = n\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}, \, s\right) \ , \tag{5.31}$$

which when integrated over x and y gives the linear particle density λ. Corresponding to this distribution, the static electric field in the x-direction at a fixed location s is given by

$$\mathcal{E}_x = \frac{eabx}{2\epsilon_0} \int_0^\infty \frac{n(T)}{a^2+u} \frac{du}{D(u)} \ , \tag{5.32}$$

where

$$D(u) = \sqrt{(a^2+u)(b^2+u)} \tag{5.33}$$

and

$$T = \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} \ . \tag{5.34}$$

The validity of Eq. (5.32) can be verified by computing the divergence of the electric field. We get

$$\frac{\partial \mathcal{E}_x}{\partial x} = \frac{eab}{2\epsilon_0} \int_0^\infty \frac{du}{D(u)} \left[\frac{n(T)}{a^2+u} + \frac{2x^2 n'(T)}{(a^2+u)^2} \right] \ . \tag{5.35}$$

Changing variable of integration from u to T,

$$dT = -\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} \right] du \tag{5.36}$$

and noting that

$$\frac{d \ln D(u)}{du} = \frac{1}{2}\left(\frac{1}{a^2+u} + \frac{1}{b^2+u} \right) \ , \tag{5.37}$$

we arrive at

$$\vec{\nabla}\cdot\vec{\mathcal{E}} = \frac{eab}{2\epsilon_0}\left[\int_0^\infty \frac{du}{D(u)}\, 2n(T)\frac{d\ln D(u)}{du} - \int_{u=0}^{u=\infty} dT\, \frac{2n'(T)}{D(u)}\right] , \tag{5.38}$$

The variable in the first integral can now be easily changed from u to T, and we obtain

$$\begin{aligned} \vec{\nabla}\cdot\vec{\mathcal{E}} &= \frac{eab}{\epsilon_0}\int_{u=0}^{u=\infty} dT\, \left[\frac{n(T)}{D^2}\,\frac{dD}{dT} - \frac{n'(T)}{D}\right] \\ &= -\frac{eab}{\epsilon_0}\int_{u=0}^{u=\infty} dT\, \frac{d}{dT}\left[\frac{n}{D}\right] = \frac{e}{\epsilon_0}\, n\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) , \end{aligned} \tag{5.39}$$

as required by Gauss's law. In passing, we give also the electric potential

$$\Phi(x,y) = -\frac{eab}{4\epsilon_0}\int_0^\infty \frac{du}{D(u)}\int_0^T dT'\, n(T') . \tag{5.40}$$

Now we are ready to compute $\langle x\mathcal{E}_x\rangle$ and $\langle y\mathcal{E}_y\rangle$. By definition,

$$\langle x\mathcal{E}_x\rangle = \frac{eab}{2\epsilon_0\lambda}\int_0^\infty \frac{du}{D(u)}\int_{-\infty}^\infty \frac{x^2dx}{a^2+u}\int_{-\infty}^\infty dy\, n(T)\, n\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) . \tag{5.41}$$

This suggests the change of variables x and y to the circular coordinates r and θ,

$$r\cos\theta = \frac{x}{\sqrt{a^2+u}} , \quad r\sin\theta = \frac{y}{\sqrt{b^2+u}} \longrightarrow \frac{dxdy}{D(u)} = rdrd\theta . \tag{5.42}$$

We also let

$$r'^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2} = r^2\left[\frac{a^2+u}{a^2}\,\cos^2\theta + \frac{b^2+u}{b^2}\,\sin^2\theta\right] . \tag{5.43}$$

The integration variable u is now changed to r' with

$$2r'dr' = \frac{r^2}{a^2b^2}\left(a^2\sin^2\theta + b^2\cos^2\theta\right) du , \tag{5.44}$$

with the integration limits u from 0 to ∞ changed to r to ∞. All these changes convert Eq. (5.41) to

$$\langle x\mathcal{E}_x\rangle = \frac{ea^3b^2}{2\pi\epsilon_0\lambda(a+b)}\int_0^\infty n\left(r^2\right) 2\pi r dr\int_r^\infty n\big(r'^2\big)2\pi r'dr' , \tag{5.45}$$

where the integration over θ has been performed with the help of

$$\int_0^{2\pi}\frac{\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta = \frac{2\pi}{b(a+b)} . \tag{5.46}$$

Note that the variables r and r' carry no dimension. With the new defined function

$$Q(r) = ab\int_0^r n\big(r'^2\big)2\pi r'dr' \qquad \text{with} \quad Q(\infty) = \lambda \ , \tag{5.47}$$

Eq. (5.45) can be integrated to give

$$\langle x\mathcal{E}_x\rangle = \frac{ea}{2\pi\epsilon_0\lambda(a+b)}\int_0^\infty dr\,\frac{dQ(r)}{dr}\,[\lambda - Q(r)] = \frac{ea\lambda}{4\pi\epsilon_0(a+b)} \ . \tag{5.48}$$

Since $\tilde{x} = \sqrt{\langle x^2\rangle} \propto a$ and $\tilde{y} = \sqrt{\langle y^2\rangle} \propto b$, we obtain the final rms envelope equation in two dimension:

$$\begin{aligned} \tilde{x}'' + K_x(s)\tilde{x} - \frac{E_x^2}{\tilde{x}^3} - \frac{r_0\lambda}{\gamma^3\beta^2}\frac{1}{\tilde{x}+\tilde{y}} &= 0 \ , \\ \tilde{y}'' + K_y(s)\tilde{y} - \frac{E_y^2}{\tilde{y}^3} - \frac{r_0\lambda}{\gamma^3\beta^2}\frac{1}{\tilde{x}+\tilde{y}} &= 0 \ . \end{aligned} \tag{5.49}$$

For a uniform distribution with elliptical symmetry in two dimensions, the half widths of the beam are $\hat{x} = 2\tilde{x}$ and $\hat{y} = 2\tilde{y}$. The emittance is $\epsilon_{x,y} = 4E_{x,y}$, since we also have $\hat{p}_{x,y} = 2\sqrt{\langle p_{x,y}\rangle}$. The envelope equation becomes

$$\begin{aligned} \hat{x}'' + K_x(s)\hat{x} - \frac{\epsilon_x^2}{\hat{x}^3} - \frac{4r_0\lambda}{\gamma^3\beta^2}\frac{1}{\hat{x}+\hat{y}} &= 0 \ , \\ \hat{y}'' + K_y(s)\hat{y} - \frac{\epsilon_y^2}{\hat{y}^3} - \frac{4r_0\lambda}{\gamma^3\beta^2}\frac{1}{\hat{x}+\hat{y}} &= 0 \ . \end{aligned} \tag{5.50}$$

These are just the well-known K-V equations. However, the rms envelope equations depicted in Eq. (5.49) are not restricted to the uniform K-V distribution and are valid for any distribution with elliptical symmetry.

Two comments are in order. First, the distribution with elliptical symmetry, represented by Eq. (5.31), is a very general distribution. Nearly all practical beam distributions fall into this category. Therefore, Sacherer's conclusion that $\langle x\mathcal{E}_x\rangle$ in Eq. (5.48) does not involve moment higher than second order is remarkable. Second, the rate of change of the beam emittance E_x, Eq. (5.19), depends on both $\langle x\mathcal{E}_x\rangle$ and $\langle p_x\mathcal{E}_x\rangle$, and will vanish if both of them do not involve moments higher than second order. Unfortunately, $\langle p_x\mathcal{E}_x\rangle$ does depend on moments of the beam which is higher than second order. As a result, the emittance introduced in Eq. (5.18) is time dependent and this renders the rms envelope equations not a closed system. The set of rms envelope equations is only closed when the distribution is uniform. It can be shown that the rate of increase of emittance is just proportional to the energy of the part of the space charge self-field that is nonlinear [5, 6, 7].

5.3 Collective Oscillations of Beams

5.3.1 One Dimension

The one-dimension envelope equation for uniform beam, Eq. (5.29), contains the external focusing term $K_x(s)$, which includes both the ideal quadrupole focusing force and the gradient errors. We first eliminate the rapidly varying part of $K_x(s)$ from the envelope equation by introducing the Floquet phase advance ψ_x, which increases by 2π each revolution turn,

$$\psi_x = \int_0^s \frac{ds}{\nu_{0x}\beta_x(s)} \ , \tag{5.51}$$

where ν_{0x} is the bare tune and β_x is the betatron function defined in the absence of the space charge self-force. Next introduce the dimensionless half beam size

$$\hat{X} = \frac{\hat{x}}{\sqrt{\epsilon_x \beta_x}} \ , \tag{5.52}$$

where the full emittance ϵ_x, defined via Eq. (5.18),

$$\epsilon_x = 3\sqrt{\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2} \ , \tag{5.53}$$

is a constant of motion because the distribution is now uniform and the space charge force is therefore linear [see Eq. (5.19)]. The envelope equation for a uniform beam in one dimension now becomes (Exercise 5.1)

$$\frac{d^2\hat{X}}{d\psi_x^2} + \nu_{0x}^2\hat{X} - \frac{\nu_{0x}^2}{\hat{X}^3} - \frac{2\pi r_0\sigma}{\gamma^3\beta^2}\,\frac{\nu_{0x}^2\beta_x^{3/2}}{\sqrt{\epsilon_x}} = 0 \ . \tag{5.54}$$

The last term on the right side depends on s through the betatron function β_x. Because β_x is periodic in s or the phase advance ψ_x, we can expand it as a Fourier series. The oscillatory part is x independent and is therefore similar to the force due to dipole errors which we have studied earlier in Sec. 5.1. Since it will drive only integer resonance and we are interested in half-integer resonance only in this section, this oscillatory part is discarded. The non-oscillatory part is related to the incoherent space charge tune shift $\Delta\nu_{sc\,x}$, or (Exercise 5.2)

$$2\nu_{0x}\Delta\nu_{sc\,x} = -\frac{2\pi r_0\sigma}{\gamma^3\beta^2}\,\frac{\nu_{0x}^2\bar{\beta}_x^{3/2}}{\sqrt{\epsilon_x}} \ , \tag{5.55}$$

where $\bar{\beta}_x$ is the betatron function averaged over the Floquet phase ψ and is equal to R/ν_{0x}, with R being the radius of the accelerator ring. In terms of $\Delta\nu_{sc\,x}$, the one-dimension envelope equation now takes the simple form‡

$$\frac{d^2\hat{X}}{d\psi_x^2} + \left(\nu_{0x}^2 + 2\nu_{0x}\Delta\nu_s \cos n\psi_x\right)\hat{X} - \frac{\nu_{0x}^2}{\hat{X}^3} + 2\nu_{0x}\Delta\nu_{sc\,x} = 0 \ , \tag{5.56}$$

where we have included the part in $K(s)$ that corresponds to quadruple gradient errors as a force possessing nth harmonic and total stopband width $\Delta\nu_s$.

When space charge is absent, the *static* solution (s or ψ_x independent) of the envelope equation is just $\hat{X} = 1$. Here, *static* is just mathematically true for the normalized beam size $\hat{X}$. In fact, this solution is not physically static, because it corresponds to the beam size

$$\hat{x} = \sqrt{\epsilon_x \beta_x} \ , \tag{5.57}$$

and β_x is a function of s. We can also see how the normalization process simplifies the analysis of the envelope equation. The solution in Eq. (5.57) says nothing more than the fact that $\sqrt{\beta_x}$ is the beam radius when the beam is *matched* to the lattice. In fact, the envelope equation, Eq. (5.29), before normalizing, is the equation satisfied by $\sqrt{\beta_x}$.

In the presence of space charge, the 'static' solution becomes

$$\hat{X} = 1 + \xi_x, \tag{5.58}$$

which can be solved as a power series in

$$\Delta_x = \frac{\Delta\nu_{sc\,x}}{\nu_{0x}} \ . \tag{5.59}$$

We obtain

$$\xi_x = -\frac{\Delta_x}{2} + \frac{3\Delta_x^2}{8} + \mathcal{O}\left(\Delta_x^3\right) \ . \tag{5.60}$$

Since $\Delta\nu_{sc\,x} < 0$, the beam size is therefore larger due to the repulsive nature of the space charge force. This can be viewed as an increase in the betatron function due to space charge by

$$\beta_x \longrightarrow \frac{\beta_x \nu_{0x}}{\nu_{0x} + \Delta\nu_{sc\,x}} \ . \tag{5.61}$$

‡The incoherent space charge tune shift is negative. Many authors prefer to denote $\Delta\nu_{sc\,x}$ as the absolute value of the tune shift. In that convention, the sign in the last term on the right side of Eq. (5.56) will be positive instead.

Now we are ready to solve the envelope equation around the 'static' solution, for which we let

$$\hat{X} = 1 + \xi_x + \delta_x(\psi_x) \ . \tag{5.62}$$

Here, δ_x represents the amplitude of oscillation of the beam width about the equilibrium value $1+\xi_x$. We only need δ_x to be infinitesimal. Therefore, we perform the power series expansion according to

$$\delta_x \ll \xi_x \ll 1 \ , \tag{5.63}$$

and keep only the first order in δ_x. We also require only an infinitesimal driving force, because this is what it needs to drive a particle into instability. Thus, we will consider the width of the stopband $\Delta\nu_s/\nu_{0x}$ to be of the same order as δ_x. This consideration leads to the equation

$$\frac{d^2\delta_x}{d\psi_x^2} + \left(4\nu_{0x}^2 + 6\nu_{0x}\Delta\nu_{sc\,x}\right)\delta_x = -2\nu_{0x}\Delta\nu_s \cos n\psi_x \ . \tag{5.64}$$

Thus the beam envelope oscillates with the natural coherent tune $2\big(\nu_{0x} + \frac{3}{4}\Delta\nu_{sc\,x}\big)$, and resonance occurs when

$$n^2 = 4\nu_{0x}^2 + 6\nu_{0x}\Delta\nu_{sc\,x} \qquad \text{or} \qquad \frac{n}{2} \approx \nu_{0x} - \frac{3}{4}\left|\Delta\nu_{sc\,x}\right| = \nu_{x\,\mathrm{incoh}} + \frac{1}{4}\left|\Delta\nu_{sc\,x}\right| \ . \tag{5.65}$$

The incoherent tune $\nu_{x\,\mathrm{incoh}} = \nu_{0x} + \Delta\nu_{sc\,x}$ can therefore be depressed beyond the half-integer $\frac{n}{2}$ by $\frac{1}{4}|\Delta\nu_{sc\,x}|$, a quarter of the incoherent tune shift before hitting the resonance as is illustrated in Fig. 5.1. Solution of Eq. (5.64) gives

$$\hat{X} = 1 - \frac{\Delta\nu_{sc\,x}}{2\nu_{0x}} - \frac{2\nu_{0x}\Delta\nu_s\cos n\psi_x}{4\nu_{0x}^2 + 6\nu_{0x}\Delta\nu_{sc\,x} - n^2} \ , \tag{5.66}$$

where only the lowest order of $\Delta\nu_{sc\,x}/\nu_{0x}$ has been included. Clearly, this solution reflects the resonance depicted in Eq. (5.65), although the solution is perturbative and is not valid near the resonance. We also see the beam envelope oscillate and that represents a quadrupole breathing mode, which is a coherent mode or collective mode because all beam particles have to participate collectively to produce this pattern of motion. This is in contrast to the incoherent motion, where a single beam particle executes betatron oscillations regardless of what the rest are doing.

Now we are in the position to study whether the force due to quadrupole errors will drive a single particle unstable at the half-integer resonance. Let us return to Eq. (5.12), the equation of motion of a single particle, which we rewrite as

$$\frac{d^2X}{d\psi_x^2} + \left(\nu_{0x}^2 + 2\nu_{0x}\Delta\nu_s\cos n\psi_x\right)X + 2\nu_{0x}\Delta\nu_{sc\,x}\frac{X}{\hat{X}} = 0 \ . \tag{5.67}$$

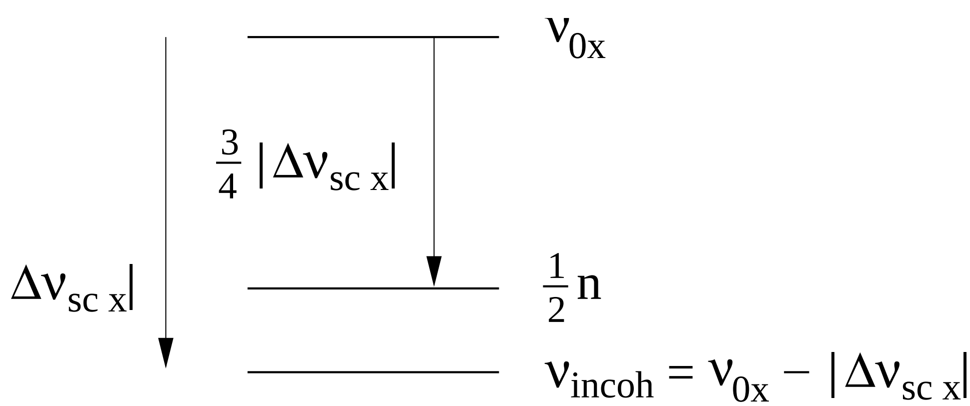

Figure 5.1: Plot showing that the incoherent tune of a one-dimensional beam, $\nu_{\text{incoh}} = \nu_{0x} - |\Delta\nu_{sc\,x}|$, can be depressed to pass the half-integer $\frac{1}{2}n$ before the coherent quadrupole tune $\nu_{0x} - \frac{3}{4}|\Delta\nu_{sc\,x}|$ reaches the half-integer instability.

where X and $\hat{X}$ are, respectively, the x-coordinate of the particle and the beam half width normalized by $\sqrt{\beta_x\epsilon_x}$. $\Delta\nu_{sc\,x}$, as given by Eq. (5.55), is the commonly quoted incoherent space charge tune shift without consideration of the beam being driven by the gradient errors of the quadrupole. The correct incoherent space charge tune shift is actually given by $\Delta\nu_{sc\,x}/\hat{X}$ (see Exercise 5.2). Since we are not interested in the rigid motion of the beam, the beam center $\langle X\rangle$ can be set to zero. When the perturbative solution $\hat{X}$ of the beam envelope in Eq. (5.66) is substituted, Eq. (5.67) becomes

$$\frac{d^2X}{d\psi_x^2} + (\nu_{0x} + \Delta\nu_{sc\,x})^2 X + 2\nu_{0x}\Delta\nu_s \cos n\psi_x \left[1 + \frac{2\nu_{0x}\Delta\nu_{sc\,x}}{4\nu_{0x}^2 + 6\nu_{0x}\Delta\nu_{sc\,x} - n^2}\right] X = 0 \ . \quad (5.68)$$

where the non-resonant free oscillations have not been included. At the particle intensity which shifts the betatron tune to half-integer, namely $\nu_{0x} + \Delta\nu_{sc\,x} = n/2$, the two terms inside the square brackets cancel, and the single-particle equation of motion reduces to

$$\frac{d^2X}{d\psi_x^2} + \left(\frac{n}{2}\right)^2 X = 0 \ . \quad (5.69)$$

We see that when the incoherent tune of a particle is shifted to half-integer, the driving force due to gradient errors cancels exactly. Thus, no resonance occurs for the particle.

The above proof appears to be overly approximated[§]. The reader can pursuit this proof to another order of the incoherent tune shift.

[§] The more accurate condition for envelope instability is $\nu_{0x}^2 - \frac{3}{2}|\nu_{0x}\Delta\nu_{sc\,x}| = \left(\frac{n}{2}\right)^2$. So the more accurate condition for "incoherent resonance" is $\nu_{0x}^2 - 2|\nu_{0x}\Delta\nu_{sc\,x}| = \left(\frac{n}{2}\right)^2$. Use of these conditions make the driving term vanish to a more accurate degree.

5.3.2 Two Dimensions

Similar to the one-dimensional case, we normalize the two-dimensional envelope equations with uniformly distributed elliptic cross section in the same way by introducing the phase advances

$$\psi_x = \int_0^s \frac{ds}{\nu_{0x}\beta_x(s)} \qquad \text{and} \qquad \psi_y = \int_0^s \frac{ds}{\nu_{0y}\beta_y(s)} \ , \tag{5.70}$$

and the dimensionless half beam radii

$$\hat{X} = \frac{\hat{x}}{\sqrt{\epsilon_x\beta_x}} \qquad \text{and} \qquad \hat{Y} = \frac{\hat{y}}{\sqrt{\epsilon_y\beta_y}} \ , \tag{5.71}$$

where ν_{0x} and ν_{0y} are the bare tunes and β_x and β_x are the betatron functions in the x and y directions, respectively, defined in the absence of the space charge self-force. Equation (5.50) that governs the motion of the beam radii becomes

$$\begin{aligned} \frac{d^2\hat{X}}{d\psi_x^2} + \left(\nu_{0x}^2 + 2\nu_{0x}\Delta\nu_{sx}\cos n_x\psi_x\right)\hat{X} - \frac{\nu_{0x}^2}{\hat{X}^3} + 2\nu_{0x}\Delta\nu_{sc\,x}\,\frac{a+b}{a\hat{X}+b\hat{Y}} = 0 \ , \\ \frac{d^2\hat{Y}}{d\psi_y^2} + \left(\nu_{0y}^2 \ + \ 2\nu_{0y}\Delta\nu_{sy}\cos n_y\psi_y\right)\hat{Y} - \frac{\nu_{0y}^2}{\hat{Y}^3} + 2\nu_{0y}\Delta\nu_{sc\,y}\,\frac{a+b}{a\hat{X}+b\hat{Y}} = 0 \ , \end{aligned} \tag{5.72}$$

where $a = \sqrt{\epsilon_x\bar{\beta}_x}$ and $b = \sqrt{\epsilon_y\bar{\beta}_y}$ are the beam radii defined through the average betatron functions $\bar{\beta}_x$ and $\bar{\beta}_y$,

$$\Delta\nu_{sc\,x} = -\frac{2\lambda r_0 R^2}{\gamma^3\beta^2\nu_{0x}a(a+b)} \qquad \text{and} \qquad \Delta\nu_{sc\,y} = -\frac{2\lambda r_0 R^2}{\gamma^3\beta^2\nu_{0y}b(a+b)} \tag{5.73}$$

are the incoherent space charge tune shifts. We have also included the forces due to gradient errors at harmonics n_x and n_y.

We first solve for the *static* beam radii

$$\hat{X} = 1 + \xi_x \qquad \text{and} \qquad \hat{Y} = 1 + \xi_y \tag{5.74}$$

in terms of the incoherent tune shifts

$$\Delta_x = \frac{\Delta\nu_{sc\,x}}{\nu_{0x}} \qquad \text{and} \qquad \Delta_y = \frac{\Delta\nu_{sc\,y}}{\nu_{0y}} \ . \tag{5.75}$$

Up to second order, we get

$$\xi_x = -\frac{\Delta_x}{2} + \frac{\Delta_x^2}{4} - \frac{\Delta_x\Delta_y}{8} \qquad \text{and} \qquad \xi_y = -\frac{\Delta_y}{2} + \frac{\Delta_y^2}{4} - \frac{\Delta_x\Delta_y}{8} \ . \tag{5.76}$$

Next, the infinitesimal displacements δ_x and δ_y are included:

$$\hat{X} = 1 + \xi_x + \delta_x \qquad \text{and} \qquad \hat{Y} = 1 + \xi_y + \delta_y \ . \tag{5.77}$$

The derivation becomes very lengthy and uninteresting. For the sake of simplicity, we study the special case of a round beam with $a = b$ and obtain the equations for small amplitude oscillation:

$$\frac{d^2\delta_x}{d\psi_x^2} + (4 + 5\Delta_x)\,\nu_{0x}^2\delta_x - \nu_{0x}^2\Delta_x\delta_y = -2\nu_{0x}^2\Delta\nu_{sx}\cos n_x\psi_x \ , \tag{5.78}$$

$$\frac{d^2\delta_y}{d\psi_y^2} + (4 + 5\Delta_y)\,\nu_{0y}^2\delta_y - \nu_{0y}^2\Delta_y\delta_x = -2\nu_{0y}^2\Delta\nu_{sy}\cos n_y\psi_y \ . \tag{5.79}$$

This is just a set of driven coupled simple-harmonic oscillators. For a round beam, we expect the incoherent space charge tune shifts in the two transverse directions to be equal. The eigentunes ν can be found by solving the eigenvalues of the matrix

$$\begin{pmatrix} 4\nu_{0x}^2 + 5\nu_{0x}\Delta\nu_{sc\,x} & -\nu_{0x}\Delta\nu_{sc\,x} \\ -\nu_{0x}\Delta\nu_{sc\,x} & 4\nu_{0y}^2 + 5\nu_{0x}\Delta\nu_{sc\,x} \end{pmatrix} , \tag{5.80}$$

from which we get

$$\nu^2 = 2\left(\nu_{0x}^2 + \nu_{0y}^2\right) + 5\nu_{0x}\Delta\nu_{sc\,x} \pm \sqrt{4\left(\nu_{0x}^2 - \nu_{0y}^2\right)^2 + \left(\nu_{0x}\Delta\nu_{sc\,x}\right)^2} \ . \tag{5.81}$$

When the two bare tunes are close so that $|\nu_{0x} - \nu_{0y}| \ll |\nu_{0x}\Delta\nu_{sc\,x}|$, the two coherent tunes are

$$\nu^2 = \begin{cases} 4\bar{\nu}^2 - 4|\nu_{0x}\Delta\nu_{sc\,x}| \\ 4\bar{\nu}^2 - 6|\nu_{0x}\Delta\nu_{sc\,x}| \end{cases} \qquad \text{or} \qquad \nu \approx \begin{cases} 2\left(\bar{\nu} - \dfrac{1}{2}\,|\Delta\nu_{sc\,x}|\right) \ , \\[2ex] 2\left(\bar{\nu} - \dfrac{3}{4}\,|\Delta\nu_{sc\,x}|\right) \ , \end{cases} \tag{5.82}$$

where $2\bar{\nu}^2 = \nu_{0x}^2 + \nu_{0y}^2$. This represents that the two transverse directions are tightly coupled. The eigenfunctions are $\sim (\delta_x + \delta_y)$ for the upper solution and $\sim (\delta_x - \delta_y)$ for the lower solution. Thus, the upper solution is the symmetric breathing mode where the oscillations are in phase in both transverse directions and the tune is $\bar{\nu} - \frac{1}{2}|\Delta_{sc\,x}|$. The lower solution is the antisymmetric mode where the beam envelope oscillates out of phase in the two transverse directions with tune $\nu - \frac{3}{4}|\Delta_{sc\,x}|$.

If the tune split is large so that $|\nu_{0x} - \nu_{0y}| \gg |\nu_{0x}\Delta\nu_{sc\,x}|$, the oscillations in the two transverse directions are almost uncoupled. The envelope oscillations in the two transverse directions are just two independent oscillators. The two coherent tunes are

$$\nu^2 = \begin{cases} 4\nu_{0x}^2 - 5|\nu_{0x}\Delta\nu_{sc\,x}| \\ 4\nu_{0y}^2 - 5|\nu_{0x}\Delta\nu_{sc\,x}| \end{cases} \qquad \text{or} \qquad \nu \approx \begin{cases} 2\left(\nu_{0x} - \dfrac{5}{8}\left|\Delta\nu_{sc\,x}\right|\right) \\ 2\left(\nu_{0y} - \dfrac{5}{8}\left|\Delta\nu_{sc\,y}\right|\right) \end{cases} . \tag{5.83}$$

Let us come back to the situation of no tune split. Suppose that the bare tunes $\nu_{0x} \sim \nu_{0y} \sim \bar{\nu}$ are $\Delta\nu$ above a half-integer or integer. We try to increase the beam intensity. and the incoherent tune shift $|\Delta\nu_{sc\,x}|$ increases accordingly. We will first meet with the condition $\frac{3}{4}|\Delta\nu_{sc\,x}| = \Delta\nu$ and the antisymmetric mode becomes unstable. However, the incoherent tune, $\nu_{0x} - |\Delta\nu_{sc\,x}|$ has passed the half integer already by a factor of $\frac{4}{3}$. The symmetric mode will meet with the half-integer and become unstable much later when $|\Delta\nu_{sc\,x}| = 2\Delta\nu$.

Similar to the one-dimensional case, the oscillatory solutions for the envelope radii can be solved. When substituted back into the single-particle equations of motion, we can verify that the driving force vanishes when the incoherent equations are at half integers, showing that the incoherent motion of individual particles can have their tunes right at half-integers without instability.

Other distributions can be analyzed in the same way. Notice that, for a round beam, the space charge tune shift $\Delta\nu_{sc\,x}$ in the last term of Eq. (5.72) is

$$\Delta\nu_{sc\,x} = -\frac{Nr_0}{2\pi\gamma^3\beta^2\epsilon} = -\frac{Nr_0}{8\pi\gamma^3\beta^2\epsilon_{\rm rms}} , \tag{5.84}$$

where $N = 2\pi R\lambda$ is the total number of particles in the beam, ϵ is the full emittance of the uniform distributed beam and $\epsilon_{\rm rms}$ is the rms emittance. Now rewrite Eq. (5.84) as

$$\Delta\nu_{sc\,x} = \frac{1}{2}\left[-\frac{Nr_0}{4\pi\gamma^3\beta^2\epsilon_{\rm rms}}\right] , \tag{5.85}$$

where the square-bracketed term is the maximum incoherent space charge tune shift of a bi-Gaussian distributed round beam. Thus what we need to remember is that the factor $\Delta\nu_{sc\,x}$ in the envelope equation represents one half of the maximum incoherent space charge tune shift for bi-Gaussian distribution. We mentioned before that for the case of strong coupling, the tune depression of the antisymmetric mode is $\frac{3}{4}|\Delta\nu_{scx}|$ and the incoherent tune shift can exceed that needed for coincidence with a half integer resonance by a factor of $\frac{4}{3}$. Now for the case of the bi-Gaussian distribution, the tune depression of this mode becomes $\frac{3}{4} \times \frac{1}{2}$ of the maximum incoherent space charge tune shift for the bi-Gaussian distributed beam, and therefore the incoherent tune can exceed that needed for coincidence with a half-integer resonance by as much as a factor of $\frac{8}{3}$. For this reason, we define a parameter G, such that Eq. (5.84) can be written as

$$\Delta\nu_{scx} = \frac{1}{G}\left[\begin{array}{c}\text{max incoherent}\\ \text{sp ch tune shift}\end{array}\right] . \tag{5.86}$$

Then, the incoherent space charge tune shift for the distribution considered will exceed the tune depression of a particular collective quadrupole mode G times better than if the distribution is uniform.

If we neglect the time dependency of the emittances, the rms envelope equations, Eq. (5.49), say that the space charge effects of all beams are the same if they have the same rms widths and emittances. These beams are called *equivalent* beams. For example, an *equivalent uniform beam* implies that the beam has the same rms dimensions as a uniform beam.

5.4 Simulations

5.4.1 One Dimension

Baartman [1] performed simulations with up to 50,000 particles according to the equation of motion:

$$x'' + \nu_0^2 x = \alpha x^{m-1}\cos(n\theta) + F_{sc} . \tag{5.87}$$

Here, the driving force leads to resonances whenever the tune ν satisfies $m\nu = n$. The space charge self-force F_{sc} on a particular particle in the simulations is simply equal to an intensity parameter multiplied by the difference between the number of particles to its left and to its right.

For a sextupole force ($m = 3$) and bare tune equals $\nu_0 = 2.45$, the relevant resonance is at $n/m = 7/3 = 2.3333$. We expect to see the beam in resonance when the coherent tune $\nu_{\rm coh} = \nu_0 - C_{33}|\Delta\nu_{sc}| = 7/3$, where $\Delta\nu_{sc}$ is the incoherent space charge tune shift

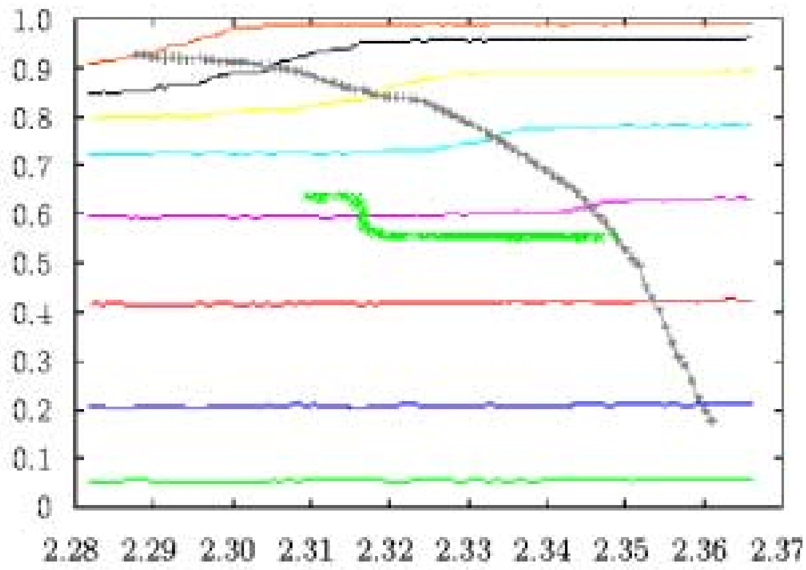

Figure 5.2: (color) Plot of the rms size (thick curve at center) of the simulated one-dimensional beam of Gaussian distribution as a function of the incoherent tune, which is used here as a measure of the beam intensity. Obviously, there is no effect on the beam when the incoherent tune crosses the 7/3 resonance. But the rms beam size increases very suddenly when the incoherent tune reaches 2.3167 corresponding to the 7/3 resonance of the coherent tune. See text for the other curves.

and $C_{33} = 7/8$ by solving the envelope equation in one dimension. This corresponds to an incoherent space charge tune shift of $|\Delta\nu_{sc}| = (2.45 - 2.333)/C_{33} = 0.1334$ or the incoherent tune of $2.45 - 0.1334 = 2.3167$. The simulations were performed for a beam with transverse Gaussian distribution. The results are plotted in Fig. 5.2 as the fraction of particles inside a given betatron amplitude versus the incoherent tune of the stationary beam of the same rms size. The incoherent tune is chosen because it serves as a measure of the beam intensity. Larger incoherent tune implies lower beam intensity. The thick curve in the center is the rms beam size. We clearly see that it passes the incoherent tune of 7/3 with nothing happening. However, there is a sharp threshold at the expected incoherent tune 2.3167. This verifies the fact that it is the coherent tune but not the incoherent tune that determines the arrival of a resonance. The horizontal curves in the figure represent the fraction of particles inside a fixed emittance for the Gaussian distribution. They step downwards as particles are driven to larger amplitudes. The stepdown occurs when a horizontal curve meets the curve connecting the + symbols. These + symbol represent the emittance at which the incoherent tune is on resonance. If we examine the figure more closely, we find that only those horizontal curves representing more than 50% of particles step downwards, and also the stepdowns are more appreciable only when the particle amplitude becomes larger. This phenomenon happens because of some halo particles residing at the very edge of the beam. They behave like a separate beam and feel the space charge force from the core of the beam as an external force. Since this is not the space charge self-force of the beam halo, our discussion of the irrelevance of the incoherent tune does not apply to these particles.

5.4.2 Two Dimensions

Machida [8] performed two-dimensional space charge simulations of the SSC Low Energy Booster by including quadrupole error forces. The horizontal bare tune was fixed at $\nu_{0x} = 11.87$ while the vertical bare tune ν_{0y} varied from 11.95 to 11.55. The maximum incoherent tune shift was kept fixed at $|\Delta\nu_{scy}| = 0.33$ with a half-integer stopband 0.02. The beam simulated had a bi-Gaussian distribution. The threshold for emittance growth was found to be roughly 11.63, when the incoherent tune had already passed the half-integer resonance of 11.50. An incoherent tune shift of 0.33 for a bi-Gaussian distributed beam is the same as an incoherent tune shift of 0.33/2=0.165 of an equivalent uniform beam. According to Eq. (5.81), the incoherent tune shift of an equivalent uniform beam is 0.199, or $2 \times 0.199 = 0.398$ for a bi-Gaussian beam. If we include the stopband, meaning that the half-integer resonance will start at $11.50 + 0.02 = 11.52$, the incoherent tune shift of an equivalent uniform beam is 0.1687, or $2 \times 0.1687 = 0.337$ for a bi-Gaussian beam. The number is very close to the incoherent tune shift of the 0.33 input into the simulations.

In other two-dimensional simulations, Machida and Ikegami [9] also demonstrated that it was the *coherent* rather than the *incoherent* tune shifts that determine the instability of a beam. Some results are illustrated in Fig. 5.3. In the simulations, the horizontal coherent quadrupole tune hits the integer 13 when the beam intensity reaches $\sim$ 15 A. We do see that the horizontal emittance increases rapidly around the beam intensity of 15 A. The vertical coherent quadrupole tune hits the integer 11 when the beam intensity is raised to around 13 to 15 A. Around those intensities, large increase in vertical emittance is evident in the plots. However, we do not see any growth of emittance when the coherent quadrupole tunes cross half integers. The simulations were performed using beams with the water-bag distribution, the K-V distribution, and the parabolic distribution. As is seen in the plots, the results do not depend much on the beam distribution.

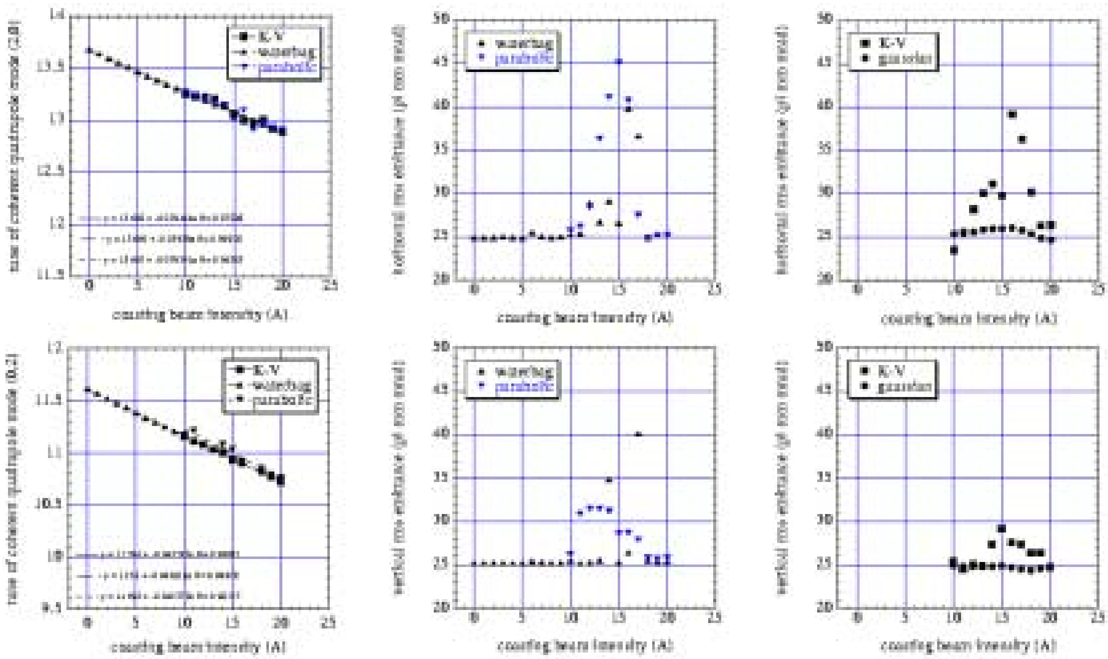

Figure 5.3: (color) Tune of coherent quadrupole mode (left) and rms emittance at 512 turns after injection (center and right) versus beam intensity. Upper figures show results in the horizontal plane while lower ones show results in the vertical plane. Rms emittance growth is observed when either the horizontal or vertical coherent quadrupole tune crosses an integer. (Reproduced from Ref. [9]).

5.5 Application to Synchrotrons

Let us apply what we have learned to some low-energy synchrotrons. For the Fermilab Booster with an injection energy of 400 MeV and round beam, the bare tunes derived from the lattice are $\nu_{0x} = 6.70$ and $\nu_{0y} = 6.80$. The nearest half-integer is 6.5. Thus, if the half-integer resonance arises from the incoherent motion of the beam particles, the largest incoherent space charge tune shift allowed will be $|\Delta\nu_{sc\,x}| = 0.20$. If the resonance comes from one of the coherent quadrupole envelope modes hitting the half-integer, the largest incoherent space charge tune shift allowed becomes¶ $|\Delta\nu_{sc\,x}|=0.296$

¶We can also make the rough estimate of assuming the two betatron bare tunes are equal, i.e., $\nu_{0x} \sim \nu_{0y} \sim 6.70$. Then the incoherent space charge tune shift according to Eq. (5.82) is $|\Delta\nu_{sc\,x}| \sim$

or $|\Delta\nu_{sc\,y}| = 0.291$. These numbers are obtained from the matrix of Eq. (5.80) by substituting $\frac{1}{2}\nu = 6.5$ for the eigentune and solving for $|\Delta\nu_{sc\,x}|$. On the other hand, from the experimentally measured beam size, the calculated incoherent space charge tune shift is 0.40, which definitely exceeds the result from incoherent motion and agrees more or less with the result from the coherent mode. So far the estimation has been based on uniform distribution. If the distribution were bi-Gaussian, the largest incoherent space charge tune shift allowed would become $|\Delta\nu_{sc\,x}| = 2\times 0.296 = 0.592$ or $|\Delta\nu_{sc\,y}| = 2\times 0.291 = 0.582$ instead for particles at the center of the beam with small amplitude betatron oscillations.

Similar computations are performed for various low-energy synchrotrons, for which the beams are mostly round and the distribution uniform. The results are tabulated in Table 5.2. We see that for all the synchrotrons listed, the space charge tune shifts computed from experimentally measured beam sizes exceed those derived from incoherent particle motion and are close to those derived from the coherent modes.

Table 5.2: Estimated inocherent space charge tune shifts for various low-energy synchrotrons. The incoherent space charge tune shifts are derived from the experimentally measured beam size (3rd column), the assumption that the half-integer resonance comes from the incoherent motion of the beam particles (4th column), and the assumption that the half-integer resonance comes from a coherent envelope mode (5th column). We see that the values from experiments exceed those from incoherent motion and agree mostly with those from the coherent modes.

Synchrotron	Bare tunes ν_{0x}/ν_{0y}	$\lvert\Delta\nu_{sc\,x}\rvert/\lvert\Delta\nu_{sc\,y}\rvert$ from experiment	from incoh motion	from coherent motion
KEK Booster	2.17/2.30	0.23	0.17	0.25/0.24
FNAL Booster	6.70/6.80	0.40	0.20	0.30/0.29
ISIS	3.70/4.20	0.40	0.20	0.31/0.27
AGS	8.75/8.75	0.58	0.25	0.33/0.33
AGS Booster	4.80/8.75	0.50	0.30	0.46/0.25
CERN PS	6.22/6.22	0.27	0.22	0.29/0.29
CERN PS-2	6.22/6.28	0.36	0.22	0.31/0.31

$|\Delta\nu_{sc\,x}| \sim \frac{4}{3}\times 0.2 = 0.267.$

5.6 Exercises

5.1. Supply the missing steps in transforming the one-dimension envelope equation from Eq. (5.30) to the normalized form of Eq. (5.54). You may need the definition of the betatron function

$$\frac{\beta_x \beta_x''}{2} - \frac{\beta_x'^{\,2}}{4} + \beta_x^2 K_x(s) - 1 = 0 \; , \tag{5.88}$$

where the prime denotes derivative with respect to s, the distance along the accelerator ring, and $K_x(s)$ is the focusing strength of the external quadrupoles.

5.2. Show that the incoherent space charge tune shift $\Delta\nu_{sc\,x}$ of a one-dimension beam uniformly distributed in the x direction and infinite in the y and s directions is given by

$$2\nu_{0x}\Delta\nu_{sc\,x} = \frac{2\pi r_0 \sigma R^2}{\gamma^3\beta^2\hat{x}} \; , \tag{5.89}$$

where the beam has extent between $\pm\hat{x}$, σ is the particle density per unit area in the y-s plane, r_0 is the classical particle radius, γ and β are the Lorentz parameters, and R is the mean radius of the accelerator ring.

5.3. Verified the expression for $\langle x\mathcal{E}_x\rangle$ given by Eq. (5.48) by computing this quantity for a round beam with (1) uniform distribution and (2) bi-Gaussian distribution.

5.4. Derive the incoherent space charge tune shifts for the various synchrotrons listed in the last column of Table 5.2 when the intensity of the beam having uniform distribution is increased so that the first coherent envelope mode reaches the half-integer resonance.

Bibliography

[1] R. Baartman, *Betatron Resonances with Space Charge*, Proceedings of Int. Workshop on Emittance in Circular Accelerators Nov. 1994, KEK, Japan, KEK report 95-7, p.273; R. Baartman, *Betatron Resonances with Space Charge*, Proceedings of Workshop on Space Charge Physics in High Intensity Hadron Rings, p.73, Ed. Luccio, A.U., and Weng, W.T., (Shelter Island, New York, May 4-7, 1998).

[2] W.T. Weng, *Space Charge Effects — Tune Shifts and Resonances*, AIP Conf. Proceedings **153** 1987, p.43..

[3] I.M. Kapchinsky and V.V. Vladimirsky, *Limitations of Proton Beam Current in a Strong Focusing Linenar Accelerator Associated with the Beam Space Charge*, Proc. Int. Conf. on High Energy Acc., CERN, 1959, p.274.

[4] F. Sacherer, *RMS Envelope Equations with Space Charge*, IEEE Trans. Nucl. Sci. **NS-18**, (PAC 1971) 1105 (1971). See also the longer report of the same title in CERN-SI-Int.-DL/70-12, Nov., 1970.

[5] P.M. Lapostolle, *Possible Emittance Increase Through Filamentation due to Space Charge*, IEEE Trans. Nucl. Sci. **NS-20**, Pac'71, 1101 (1971).

[6] T.P. Wangler, K.R. Crandall, R.S. Mills, and M. Reiser, *Relation Between Field Energy and RMS Emittance in Intense Particle Beams*, Proc. Particle Accelerator Conference 1985, Vancouver, BC, p.2196.

[7] I. Hoffmann, *Space Charge Dominated Beams Transport*, CERN Accelerator School for Advanced Accelerator Physics, Oxford, England, September 16027, 1985, p.327.

[8] S. Machida, *Space Charge Effects in Low Energy Proton Synchrotrons*, Nucl. Inst. Meth. **A309**, 43 (1991).

[9] S. Machida and M. Ikegami, *Simulation of Space Charge Effects in a Synchrotron*, Proceedings of Workshop on Space Charge Physics in High Intensity Hadron Rings, p.73, Ed. Luccio, A.U., and Weng, W.T., (Shelter Island, New York, May 4-7, 1998).